Lecturered By Dr. H.K. Yadav Date 15-05-2020.
Deptt. of Mathematics Marwari College, DBG
B.Sc-part-II (Sub) ~~Diff.~~ Calculus.

Q.1. Show that x^x is a minimum for $x = \frac{1}{e}$.

Soln: Let $y = x^x$

Taking logarithm on both sides, we get

$$\log y = \log x^x \Rightarrow \log y = x \log x$$

If the logarithm of a function is minimum, then the function is also minimum.

$\therefore$ Let $u = x \log x$ ——— (I)

Diff. w.r. to x, we get

$$\frac{du}{dx} = 1 \cdot \log x + x \cdot \frac{1}{x} = 1 + \log x \text{ ——— II}$$

Again, diff. w.r. to x, we get

$$\frac{d^2u}{dx^2} = \frac{1}{x} \text{ ——— (III)}$$

For maxima or minima, $\frac{du}{dx} = 0$

i.e. by II $1 + \log x = 0 \Rightarrow \log x = -1 = \log e^{-1}$

$$x = \frac{1}{e}$$

From III, $\frac{d^2u}{dx^2} = e$ as $x = \frac{1}{e}$

$\therefore$ u is minimum as $\frac{d^2u}{dx^2} > 0$.

Hence, the given function y is also minimum when $x = \frac{1}{e}$.

Q.2. If $y = \frac{x}{\log x}$, show that $x = e$ gives minimum value of y.

Soln: Since, $y = \frac{x}{\log x}$

Diff. w.r. to x, we get

$$\frac{dy}{dx} = \frac{\log x - x \cdot \frac{1}{x}}{(\log x)^2} = \frac{\log x - 1}{(\log x)^2} \text{ ——— (I)}$$

For max. or min. of y, $\frac{dy}{dx} = 0 \Rightarrow \frac{\log x - 1}{(\log x)^2} = 0 \Rightarrow \log x - 1 = 0$
$\Rightarrow \log x = 1$
$\therefore x = e$.

Again, diff. ① w.r.t. x, we get

$$\frac{d^2y}{dx^2} = \frac{\frac{1}{x}(\log x)^2 - (\log x - 1)\, 2(\log x)\frac{1}{x}}{(\log x)^4}$$

When $x = e$, $\frac{d^2y}{dx^2} = \frac{\frac{1}{e}}{1} = \frac{1}{e} > 0$.

$\therefore$ y is min. when $x = e$. proved.

Q.3. Find the maximum value of $x^p y^q$ when $x + y = a$.

Soln Let $z = x^p y^q = x^p (a-x)^q$ $[\because y = a - x]$

Diff. w.r.t. x, we get

$$\frac{dz}{dx} = p x^{p-1}(a-x)^q - q x^p (a-x)^{q-1}$$

$$= x^{p-1}(a-x)^{q-1}[p(a-x) - qx] \quad \text{——— ①}$$

$\therefore$ For max. or min. of z, $x^{p-1}(a-x)^{q-1}[p(a-x) - qx] = 0$

$\therefore$ $x = 0$, $x = a$, $x = \frac{ap}{p+q}$ if $p > 1, q > 1$

Again, diff. I w.r.t. x, we get

$$\frac{d^2z}{dx^2} = (p-1)x^{p-2}(a-x)^{q-1}[p(a-x) - qx]$$
$$-(q-1)x^{p-1}(a-x)^{q-2}[p(a-x) - qx] + x^{p-1}(a-x)^{q-1}[-p-q]$$

When $x = 0$, $x = a$, $\frac{d^2z}{dx^2} = 0$

When $x = \frac{ap}{p+q}$, $\frac{d^2z}{dx^2} = \left(\frac{ap}{p+q}\right)^{p-1}\left(a - \frac{ap}{p+q}\right)^{q-1}(-p-q) < 0$.

$\therefore$ z is max. when $x = \frac{ap}{p+q}$

$\therefore$ Maximum value of $z = \left(\frac{ap}{p+q}\right)^p \left(a - \frac{ap}{p+q}\right)^q$

$$= \left(\frac{ap}{p+q}\right)^p \left(\frac{aq}{p+q}\right)^q$$

$$= \frac{a^{p+q}}{(p+q)^{p+q}} p^p q^q$$

Ans.